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NONSINGULAR DEFORMATIONS OF SPACES WITH NORMAL CROSSINGS. I

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Introduction

We wish to study one-parameter families of compact complex spaces. We will describe a certain construction which can be performed on a class of complex spaces and which will yield a topological manifold homeomorphic to any nonsingular fibre of a one-parameter family containing the given complex space as singular fibre (as long as the given structure sheaf on the complex space is the same as the one which is induced as a fibre of a family). The structure sheaf plays a nontrivial role since, for example, if $\pi: \mathscr{M} \to \varDelta = \{z \mid |z| < 1, z \in \boldsymbol{C}^1\}$ is a one-parameter family $M_t = \pi^{-1}(t)$ such that with respect to appropriate local coordinates $\pi(w_1, \cdots, w_n) = w_1^k$, then M_t for $t \neq 0$ is a k-sheeted covering of M_0 which in general will be topologically distinct from M_0 even though M_0 is a nonsingular submanifold of $\mathscr{M}$.

The class of complex spaces which we study is the one with normal crossing singularities. For these spaces we will give a simple condition which must be satisfied if they are to belong to a one-parameter family. For those spaces which are members of a one-parameter family we will show how they determine the topology of the nonsingular fibres.

0. Basic definitions and assumptions

We remind the reader of some standard definitions (consult, for example, Grauert and Kerner [2]). Let X be a topological space, and $\mathscr{A}$ a sheaf of local complex algebras on X. We suppose that the unit $1_x \in \mathscr{A}_x$ varies continuously with $x \in X$. If $\mathscr{M}_x$ is the maximal ideal of $\mathscr{A}_x$, then $\mathscr{A}_x/\mathscr{M}_x$ is isomorphic to $\boldsymbol{C}$ under the isomorphism which sends $1_x + \mathscr{M}_x$ into $1 \in \boldsymbol{C}$. The pair $(X, \mathscr{A})$ is called a *complex ringed space*, and $\mathscr{A}$ the *structure sheaf*.

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two complex ringed spaces. By a *morphism* $\Phi: (X, \mathscr{A}) \to (Y, \mathscr{B})$ we mean a pair of continuous maps (φ, φ^*), where φ maps X into Y, and φ^* maps the sheaf $\varphi^{-1}\mathscr{B} = \{(x, b) \mid x \in X, b \in \mathscr{B}_{\varphi(x)}\}$ into $\mathscr{A}$ so that φ^* is a sheaf map which is a homomorphism of local complex algebras on each stalk. Φ is a *bimorphism* if there is a morphism $\Psi = (\psi, \psi^*): (Y, \mathscr{B})$

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$\to (X, \mathscr{A})$ such that $\Phi \circ \Psi =$ identity and $\Psi \circ \Phi =$ identity where the notation is obvious.

Let G be an open region in C^n, and A an analytic set in G. Let $\mathscr{O}(G)$ be the sheaf of germs of holomorphic functions on G, and $\mathscr{I} \subseteq \mathscr{O}(G)$ be a *coherent* sheaf of ideals $\mathscr{I}_x$ such that $A = \{x \in G \mid \mathscr{I}_x \neq \mathscr{O}_x(G)\}$. Then the stalks of the sheaf $\mathscr{O}(G)/\mathscr{I}$ are zero outside of A, and therefore $\mathscr{H} = \mathscr{O}(G)/\mathscr{I}$ can be considered to be a sheaf of local complex algebras on A. Hence $(A, \mathscr{H})$ is a complex ringed space. Such spaces are called complex models. A complex ringed space is a *complex space* if:

(i) X is Hausdorff,

(ii) to every $x \in X$ there are a neighborhood U and a complex model $(A, \mathscr{H})$ such that $(U, \mathscr{A}|_U)$ is bimorphic to $(A, \mathscr{H})$. One can easily see that the complex spaces form a category. A morphism of complex spaces will be called a *holomorphic mapping*.

Let A_λ be an analytic set in a region G_λ of C^{n_λ}, where $\lambda = 1$ or 2. Suppose A_λ has structure sheaf $\mathscr{H}_\lambda$ where $\mathscr{H}_\lambda = \mathscr{O}(G_\lambda)/\mathscr{I}_\lambda$ as in the definition of a complex model. Let $\psi: G_1 \to G_2$ be a holomorphic mapping in the classical sense such that $\psi(A_1) \subset A_2$. Then the inverse image $\psi^{-1}\mathscr{O}(G_2)$ is contained in $\mathscr{O}(G_1)$. Suppose $\psi^{-1}\mathscr{I}_2 \subset \mathscr{I}_1$. Then by passing to quotients we can define a sheaf map $\varphi^*: \psi^{-1}(\mathscr{H}_2) \to \mathscr{H}_1$. If we set $\psi | A_1 = \varphi$, we have defined a morphism $(\varphi, \varphi^*): (A_1, \mathscr{H}_1) \to (A_2, \mathscr{H}_2)$. Such a morphism will be said to be *generated* by ψ. The following result is well-known (see e.g. Grauert [1]).

Proposition 1. *Let $(\varphi, \varphi^*): (A_1, \mathscr{H}_1) \to (A_2, \mathscr{H}_2)$ be a morphism of complex spaces. Let $x \in A_1$ and $y = \varphi(x) \in A_2$. Then there are complex models $(M_1, \mathscr{A}_1)$, $(M_2, \mathscr{A}_2)$ with $M_1 \subseteq G_1 \subseteq C^{n_1}$, $M_2 \subseteq G_2 \subseteq C^{n_2}$ where G_1 and G_2 are regions in C^{n_1} and C^{n_2}, and there are neighborhoods U of x and V of y and bimorphisms $(\psi_1, \psi_1^*): (M_1, \mathscr{A}_1) \to (U, \mathscr{H}_1|_U)$, $(\psi_2, \psi_2^*): (M_2, \mathscr{A}_2) \to (V, \mathscr{H}_2|_V)$ such that the composition $(\psi_2, \psi_2^*)^{-1}(\varphi, \varphi^*)(\psi_1, \psi_1^*)$ is generated by a holomorphic map $\psi: G_1 \to G_2$.*

Let $(X, \mathscr{H})$ be a complex space, and $(M, \mathscr{O})$ a complex manifold where $\mathscr{O}$ is the sheaf of germs of holomorphic functions on M. Suppose $(\pi, \pi^*): (X, \mathscr{H}) \to (M, \mathscr{O})$ is a holomorphic map. For $t \in M$, $X_t = \pi^{-1}(t)$ is an analytic subset of X. Let $\mathscr{M}_t \subseteq \mathscr{O}_t$ be the maximal ideal of $\mathscr{O}_t$, and $\mathscr{M}_t \circ \pi$ the ideal of $\mathscr{H}$ generated by $\pi^*(\pi^{-1}\mathscr{M}_t)$. If one defines $\mathscr{H}_t = \mathscr{H} / \mathscr{M}_t \circ \pi$, then $\mathscr{H}_t$ vanishes outside of X_t, and $(X_t, \mathscr{H}_t)$ is a complex space. The morphism $(\pi, \pi^*): (X, \mathscr{H}) \to (M, \mathscr{O})$ hence defines a *family of complex spaces* $(X_t, \mathscr{H}_t)$. We abbreviate this notation to $\pi: X \to M$.

Now let $\mathscr{M}$ be a complex manifold, and ω be a *proper* holomorphic map of $\mathscr{M}$ *onto* $\Delta = \{z \mid |z| < 1, z \in C\}$, the unit disk in C. Let $\mathscr{A}$ and $\mathscr{O}$ be the sheaves of germs of holomorphic functions on Δ and $\mathscr{M}$ respectively. If we give each fibre $M_t = \omega^{-1}(t)$ the structure sheaf described above, then we say that the triple $(\mathscr{M}, \omega, \Delta)$ is a *one-parameter family of compact complex spaces*. For simplicity, in subsequent work we will always assume $\mathscr{M}$ and M_t to be connected for all $t \in \Delta$. By a *one-parameter family* we shall always mean a one-parameter family of connected compact complex spaces. We have

Proposition 2. *Let $(\mathcal{M}, \omega, \Delta)$ be a one-parameter family. Then any general member $(M_t, \mathcal{H}_t)$ is a nonsingular complex manifold with reduced structure sheaf. By this we mean that the set of points $t \in \Delta$, for which $(M_t, \mathcal{H}_t)$ is not a complex manifold with $\mathcal{H}_t$ the sheaf of germs of holomorphic functions on M_t, is a set without accumulation point in Δ.*

Proof. If we are willing to use the proper mapping theorem of Remmert the proof would be immediate. We propose instead the following elementary argument. Let $\pi_*: T\mathcal{M} \to T\Delta$ be the induced map of tangent spaces. Let $A \subseteq \mathcal{M}$ be the analytic subset consisting of all points $p \in \mathcal{M}$ for which $\pi_*: T_p\mathcal{M} \to T_{\pi(p)}\Delta$ is the zero map. Let A_s be the singular points of A. Then $A - A_s$ is a complex manifold, and π is constant on each component of $A - A_s$. Thus π is constant on each irreducible component of A. Now suppose $\pi(A)$ has an accumulation point in Δ. Call it s. Then there is a sequence of distinct points $\{t_\nu\} \subseteq \pi(A)$ for which $t_\nu \to s$. Since π is proper (by passing to a subsequence if necessary) we find a sequence $p_\nu \in A$, $p \in \mathcal{M}$ such that $\pi(p_\nu) = t_\nu$, $\pi(p) = s$ and $p_\nu \to p$. But then π_* is zero on $T_p\mathcal{M}$, so $p \in A$. Since the t_ν are distinct, each p_ν belongs to a different irreducible component of A. Thus we conclude that any neighborhood of the point $p \in A$ has infinitely many irreducible components. This is impossible for an analytic set (see, for example, Gunning and Rossi [3, pp. 89, 116]). q.e.d.

We will refer to this proposition as Bertini's theorem.

Let X be a complex space. We wish to define what it means for X to have only *normal crossing* singularities. Let X also refer to the underlying topological space of X. Suppose $X = UX_\alpha$, $1 \leq \alpha \leq l$, where each X_α is a nonsingular complex manifold. Let $x \in X$, and let $(\alpha_1, \cdots, \alpha_p)$ be the set of integers for which $x \in X_{\alpha_i}$. Then we suppose that there are a set $\beta = (\beta_1, \cdots, \beta_p)$ of positive integers and a neighborhood of x in X, which is bimorphic to the complex model in an open disk D around the origin $(0) \in C^q$ $(q \geq p)$ defined by $z_1 \cdots z_p = 0$ with structure sheaf $\mathcal{O}(D)/(z^\beta)$, where $\mathcal{O}(D)$ is the sheaf of germs of holomorphic functions of D, and (z^β) is the ideal of $\mathcal{O}(D)$ generated by the holomorphic function $z^\beta = z_1^{\beta_1} \cdots z_p^{\beta_p}$. We also suppose that this biholomorphic map takes x into 0. If this situation holds for every $x \in X$, we say that X has only normal crossing singularities.

Now let $(\mathcal{M}, \omega, \Delta)$ be a one-parameter family. Suppose $(M_0, \mathcal{H}_0)$ is a singular fibre (i.e., M_0 singular as a complex analytic set or $\mathcal{H}_0$ not reduced). Then by Bertini's theorem there is a neighborhood of 0 such that for all points t in the neighborhood, except $0, M_t$ is nonsingular and $\mathcal{H}_t$ is reduced. We restrict our attention to the portion of $\mathcal{M}$ above this neighborhood. By changing coordinates on Δ we may assume that this small neighborhood again is $\Delta = \{z \mid |z| < 1, z \in C\}$. We denote the new family by the same symbols $(\mathcal{M}, \omega, \Delta)$, although ω has really been changed by the change of coordinates. If $(M_0, \mathcal{H}_0)$

has only normal crossing singularities, we may assume that ω is given locally by $\omega(z_1, \cdots, z_n) = z^\beta$ where $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_i \geq 0$, $\beta \in \boldsymbol{Z}$ and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. We will use this multi-index notation when convenient in order not to get lost in a cloud of indices. A connected compact complex space will be said to be *admissible*, if it has only normal crossing singularities, and there exists a one-parameter family in which it occurs as a singular fibre.

We shall give the construction of the topological nonsingular model, and prove that it is indeed homeomorphic to the nonsingular fibres of any one-parameter family in which the complex space occurs as singular fibre for a set of cases in ascending order of difficulty. We could have done only the most difficult case, since it contains all of the easier cases. We think this would have made the procedure more difficult to understand, so we have built up from the simple to the complicated. The cases are as follows. (In each case we list the "worst" possible local behavior of ω.)

Case I. $(M_0, \mathscr{H}_0)$ occurs as a singular fibre with $\omega(z) = z_1^k$, in terms of local coordinates $z = (z_1, \cdots, z_n)$.
Case II. $(M_0, \mathscr{H}_0)$ occurs as a singular fibre with $\omega(z) = z_1^{\beta_1} z_2^{\beta_2}$.
Case III. $(M_0, \mathscr{H}_1)$ occurs as a singular fibre with $\omega(z) = z_1^{\beta_1} z_2^{\beta_2} z_3^{\beta_3}$.
Case IV. The general case: $\omega(z) = z^\beta$, $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_i \in \boldsymbol{Z}$, $\beta_i \geq 0$.

1. The nonsingular model in Case I

Let $(M, \mathscr{H})$ be an admissible complex space which occurs as a singular fibre in a one-parameter family $(\mathscr{M}, \omega, \Delta)$ for which $(M, \mathscr{H}) \cong (M_0, \mathscr{H}_0)$ and for which one can find local coordinates $z = (z_1, \cdots, z_n)$ for a neighborhood in $\mathscr{M}$ of an arbitrary point in M_0 such that $\omega(z) = z_n^e$ with $e > 1$ ($\cong$ means bimorphic). Since M is connected, e will be the same for all points in M_0. We can find a finite covering $\{U_j\}$ of M with open sets such that on each set U_j, $\mathscr{H}$ is bimorphic to the sheaf $\mathscr{O}(D)/(z_{jn}^e)$, with D an open disk around 0 in $\boldsymbol{C}^n$. If $U_j \cap U_k \neq \emptyset$, then a nonvanishing holomorphic function $f_{jk}(x)$ on $U_j \cap U_k$ is defined by the relations

$$z_{jn} = F_{jk}(z_{k1}, \cdots, z_{kn}) \cdot z_{kn} ,$$
$$f_{jk}(x) = F_{jk}(z_{k1}(x), \cdots, z_{kn-1}(x), 0) .$$

It is easy to see that $\{f_{jk}\}$ defines a 1-cocycle on the nerve of the covering $\{U_j\}$ and thus gives an element $f \in H^1(M, \mathscr{O}^*)$ where $\mathscr{O}^*$ denotes the sheaf of germs of nonvanishing holomorphic functions on the complex manifold M (reduce the complex space $(M, \mathscr{H})$). In fact, since $\omega = z_{jn}^e$, $f_{jk}^e = 1$. Thus f_{jk} is an e-th root of unity. Let $\boldsymbol{Z}_e$ denote the group of e-th roots of unity (a multiplicative subgroup of $S^1 \subseteq \boldsymbol{C}$). Then $f \in H^1(M, \boldsymbol{Z}_e)$, and thus f defines an *e-sheeted unbranched covering* $\nu(M_0)$ of M_0, which is the nonsingular model for Case I.

We will show that the cohomology class f depends only on the structure sheaf $\mathscr{H}$ and not on the particular choice of family $(\mathscr{M}, \omega, \Delta)$. For let $(\mathscr{N}, \pi, \Delta)$ be any other one-parameter family for which $(M, \mathscr{H}) \cong (N_0, \mathscr{K}_0)$, where $N_0 = \pi^{-1}(0)$, $\mathscr{K}_0$ is the induced structure sheaf, and locally $\pi = w_n^e$. Let $\{U_j\}$ cover M such that locally M is given by $A_{z_j} = \{z_j \in D_z \mid z_{jn}^e = 0\}$ and $A_{w_j} = \{w_j \in D_w \mid w_{jn}^e = 0\}$, where z_j and w_j are local coordinates for $\mathscr{M}$ and $\mathscr{N}$, and D_z, D_w are open disks around $0 \in C^n$. Since $(M, \mathscr{H})$ is bimorphic to both $(M_0, \mathscr{H}_0)$ and $(N_0, \mathscr{K}_0)$, there is a bimorphism between $(A_z, \mathscr{O}(D_z)/(z_{jn}^e))$ and $(A_w, \mathscr{O}(D_w)/(w_{jn}^e))$. By Proposition 1, § 0, we may assume that this bimorphism is induced by a map $\varphi_j\colon D_z \to D_w$. It is not hard to see that this map takes the form

$$\varphi_{jn}(z_{j1}, \cdots, z_{jn}) = z_{jn} f_j(z_{j1}, \cdots, z_{jn}) ,$$

where $f_j(z_{j1}(x), \cdots, z_{jn-1}(x), 0) \neq 0$, and $\varphi_j = (\varphi_{j1}, \cdots, \varphi_{jn})$. Then it follows that $f_{jk} = f_j^{-1} \bar{f}_{jk} f_k$, where

$$w_{jk} = \bar{F}_{jk}(w_{k1}, \cdots, w_{kn}) , \qquad \bar{f}_{jk}(x) = \bar{F}_{jk}(w_{k1}(x), \cdots, w_{kn-1}(x), 0) .$$

Thus $f = \{f_{jk}\}$ and $\bar{f} = \{\bar{f}_{jk}\}$ define the same class in $H^1(M, C^*)$, so that f and $\bar{f}$ define the same class in $H^1(M, Z_e)$. Therefore the covering defined by f depends only on $\mathscr{H}$ and not on $(\mathscr{M}, \omega, \Delta)$. We denote this e-sheeted covering by $\nu(M, \mathscr{H})$, and call it the *topological nonsingular model* of $(M, \mathscr{H})$. Note that this terminology has very little to do with the same expression in algebraic geometry.

Remark. This covering could be defined as follows. The cocycle $\{f_{jk}\}$ determines a complex line bundle $[f]$ over M. Let ξ_i be a local fibre coordinate for $[f]$. Then it is easy to see that the collection of local subvarieties $\{\xi_i^e = 1\}$ fits together to form a nonsingular submanifold W of $[f]$. The fibre projection makes W into an unramified e-sheeted covering of M. Then $W = \nu(M, \mathscr{H})$.

2. The deformation theorem for Case I spaces

In this section we prove the following theorem.

Theorem. *Let $(\mathscr{M}, \omega, \Delta) = \{M_t \mid t \in \Delta\}$ be a one-parameter family. Suppose M_t is nonsingular for $t \neq 0$, and $(M_0, \mathscr{H}_0)$ is a Case* I *space. Then M_t is diffeomorphic to $(M_0, \mathscr{H}_0)$ for $t \neq 0$.*

Remark. We know two proofs of this result, and since they are both easy we give them both.

Proof 1. We suppose $\nu(M_0, \mathscr{H}_0)$ is an e-sheeted covering of M_0. Let $\bar{\Delta}$ be another copy of Δ, and $\varphi\colon \bar{\Delta} \to \Delta$ be given by $\varphi(\zeta) = \zeta^e$. Then we define a space $W \subseteq \bar{\Delta} \times \mathscr{M}$ by

$$W = \{(\zeta, z) \mid \zeta^e = \omega(z)\} .$$

Then $W = \cup W_j$ where

$$W_j = \{(\zeta, z_j) \mid \zeta^e = z_{jn}^e\} ,$$

and $z_j = (z_{j1}, \cdots, z_{jn})$ is a local coordinate in $\mathcal{M}$ around a point of M_0. Here $n = \dim \mathcal{M}$, and we are assuming that the domains of these local coordinates cover $\mathcal{M}$ (if not, we could just take $\mathcal{M}$ to be a little smaller). W_j itself is a union $W_j = \bigcup_{l=1}^{e} W_j^l$ with

$$W_j^l = \{(\zeta, z) \mid \rho^l \zeta = z_{jn}\} ,$$

where $\rho = \exp(2\pi i/e)$. We construct a new manifold $\bar{\mathcal{M}}$ as follows. W_j is a union of e-sheets which intersect along a portion of M_0. We separate these, and consider $\bar{W}_j = \bigcup_{l=1}^{e} W_j^l$ as a *disjoint* union. Then $\bar{\mathcal{M}} = \bigcup \bar{W}_j$ where we make the following identifications. As in § 1 we have

$$z_{jn} = F_{jk}(z_k) \cdot z_{kn} \qquad \text{on } V_j \cap V_k$$

with $F_{jk}^e = 1$, where V_j is the domain of z_j. We also have

$$z_j' = G_{jk}(z_k) \qquad \text{on } V_j \cap V_k ,$$

where $z_j' = (z_{j1}, \cdots, z_{jn-1})$. We see that F_{jk} is locally constant, and is an e-th root of unity. As before we set $f_{jk} = F_{jk}(z_k', 0)$. We identify $(\zeta, z_j) \in W_j^l$ with $(\zeta, z_k) \in W_k^m$ if

$$z_j' = G_{jk}(z_k) , \quad z_{jn} = F_{jk}(z_k) \cdot z_{kn} , \quad \rho^l = \rho^m \cdot f_{jk} .$$

This makes $\bar{\mathcal{M}}$ into a complex manifold. We have a natural projection $\bar{\pi}\colon \bar{\mathcal{M}} \to \bar{\Delta}$ and a commuting diagram

$$\begin{array}{ccc} \bar{\mathcal{M}} & \longrightarrow & \mathcal{M} \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \omega} \\ \bar{\Delta} & \underset{\varphi}{\longrightarrow} & \Delta \end{array} .$$

It is easy to see that the differential of $\bar{\pi}$ has rank 1, and thus each fibre is a nonsingular complex manifold. In fact $\bar{\pi}$ is proper, and thus $\bar{M}_0$ is diffeomorphic to $\bar{M}_t$ for $t \in \bar{\Delta}$. If $t \neq 0$, then $\bar{M}_t = M_t$, and if $t = 0$, then $\bar{M}_0 = \nu(M_0, \mathcal{H}_0)$. Hence M_t is diffeomorphic to $\nu(M_0, \mathcal{H}_0)$.

Proof 2. Cover M_0 with a finite number of open sets W_j which are domains of coordinate charts z_j for $\mathcal{M}$. We assume $W_j = \{x \mid \sup_m |z_{jm}(x)| < 1\}$ where $z_j = (z_{j1}, \cdots, z_{jn})$. On W_j we assume $\omega(x) = z_{jn}^e(x) = z_{jn}^e$ (we omit x). Let $z_j' = (z_{j1}, \cdots, z_{jn-1})$. Then on $W_j \cap W_k$ we have

$$z_{jn} = F_{jk}(z_k) \cdot z_{kn} , \qquad z_j' = G_{jk}(z_k', z_{kn}) .$$

Since $z_{jn}^e = z_{kn}^e$, we conclude $F_{jk}^e = 1$. Thus $|F_{jk}| = 1$, and $|z_{jn}| = |z_{kn}|$ on

$W_j \cap W_k$. Let $r = |z_{jn}|$ on W_j. Then we get a continuous function r defined on $\cup W_j = W$.

Let $\Delta^+ = \{t \mid 0 \leq t < 1\} \subseteq \Delta$, $\mathcal{M}^+ = \omega^{-1}(\Delta^+)$, and $z_{jn} = r\theta_j$ with $|\theta_j| = 1$. Then $z_{jn}^e \geq 0$, on $\mathcal{M}^+$ implies $\theta_j^e = 1$. Thus we can write $W_j^+ = W_j \cap \mathcal{M}^+$ in the form

$$W_j^+ = \left\{(z_j', r\theta_j) \mid 0 \leq r < 1,\ \theta_j^e = 1,\ \sup_{m<n} |z_{jm}| < 1\right\} .$$

(Here, as elsewhere, we identify W_j with its image under the chart map z_j). Let $W^+ = \bigcup_j W_j^+$. Then M_0 is the subspace defined by $r = 0$. The points $(z_j', r\theta_j)$ and $(z_k', r\theta_k)$ are identified if and only if

$$(2) \qquad \theta_j = F_{jk}(z_k', r\theta_k)\cdot\theta_k , \qquad z_j' = G_{jk}(z_k', r\theta_k) .$$

That is, $(z_j', r\theta_j)$ and $(z_k', r\theta_k)$ are identified if and only if they define the same point of $W_j^+ \cap W_k^+$. We define $\nu(W_j^+)$ by

$$(W_j^+) = \{(z_j', r, \theta_j) \mid 0 \leq r < 1,\ \theta_j \in S^1,\ \theta_j^e = 1,\ |z_j'| < 1\} ,$$

where S^1 is the unit circle in S^1, and $|z_j'| = \sup_{m<n} |z_{jm}|$. Then we form a union $\bigcup_j \nu(W_j^+) = \nu(W^+)$ by identifying $(z_j', r, \theta_j) \in \nu(W_j^+)$ with $(z_k', r, \theta_k) \in \nu(W_k^+)$ if and only if

$$\theta_j = F_{jk}(z_k', r\theta_k)\theta_k , \qquad z_j' = G_{jk}(z_k', r\theta_k) ,$$

where $(z_k', r\theta_k)$ defines a point of $W_k^+ \cap W_j^+$. Thus (W_j^+) is a (disjoint) union of e copies of $\{(z_j', r) \mid 0 \leq r < 1, |z_j'| < 1\}$, and $\nu(W^+)$ is a topological manifold with boundary $B = \cup \{(z_j', \theta_j)\}$. Then μ: $(z_j', r, \theta_j) \to (z_j', r\theta_j)$ defines a continuous map from $\nu(W^+)$ onto W^+, which is a homeomorphism from $\nu(W^+) - B$ onto $W^+ - M_0$. Next we replace W^+ with $\nu(W^+)$, forming a new manifold $\nu(\mathcal{M}^+) = (\mathcal{M}^+ - W^+) \cup \nu(W^+)$. We extend μ to a continuous map μ: $\nu(\mathcal{M}^+) \to \mathcal{M}^+$ by setting it equal to the identity on $\nu(\mathcal{M}^+) - \nu(W^+)$. Then μ is a homeomorphism from $\nu(\mathcal{M}^+) - \mu^{-1}(M_0)$ onto $\mathcal{M}^+ - M_0$ where $\mu^{-1}(M_0) = B$. If we write $\mu^{-1}(M_0) = \nu(M_0)$, then it is easy to see that $\nu(M_0) = \nu(M_0, \mathcal{H}_0)$ is the topological nonsingular model.

The map $\omega\mu$: $\nu(\mathcal{M}^+) \to \Delta^+$ is continuous, and $(\omega\mu)^{-1}(t) = M_t$ for $t > 0$. For $t = 0$,

$$(\omega\mu)^{-1}(0) = \nu(M_0, \mathcal{H}_0) = \nu(M_0) .$$

It is clear that $\nu(\mathcal{M}^+) - \nu(M_0)$ is a smooth manifold. We will show that M_t is homeomorphic to $\nu(M_0, \mathcal{H}_0)$ by introducing a differentiable structure on $\nu(\mathcal{M}^+)$, which is an extension of the given differentiable structure on $\nu(\mathcal{M}^+) - \nu(M_0)$, and we will find a differentiable function on $\nu(\mathcal{M}^+)$ with no critical points, which has as level sets the manifolds $(\omega\mu)^{-1}(t) = M_t$ for $1 > t \geq 0$. Hence M_t will be homeomorphic to M_0.

If we set

$$e(q) = \begin{cases} \exp(-1/q) & \text{for } q > 0 , \\ 0 & \text{for } q = 0 , \end{cases}$$

then $e(q)$ is a smooth monotone increasing function of q. If $\boldsymbol{R}^+ = \{r \mid r \geq 0\}$, the map $r \to \tau$ defined by $e(\tau) = r^e$ is a homeomorphism from $[0, 1)$ to $\boldsymbol{R}^+$, and $r(\tau) = [e(\tau)]^{1/e}$ is a smooth function of τ. Then we introduce coordinates (z'_j, r, θ_j) on $\nu(W_j^+)$, and we have replacing (2)

$$(3) \qquad \theta_j = F_{jk}(z'_k, r(\tau)\theta_k)\cdot\theta_k , \qquad z'_j = G_{jk}(z'_k, r(\tau)\theta_k) .$$

Hence $(z'_k, \tau, \theta_k) \to (z'_j, \tau, \theta_j)$ is a smooth map. So the coordinates $(\tau, \theta_j, y_j, z_j)$ form a system of differentiable coordinates (θ_j is not really a coordinate; it is more like an index) in a neighborhood $\nu(W^+)$ of $\nu(M_0)$. It is clear that this is a continuation of the smooth structure on $\nu(\mathscr{M}^+) - \nu(M_0)$. The function τ is a smooth function on $\nu(\mathscr{M}^+)$ with no critical points, and has the manifolds $(\omega\mu)^{-1}(t)$ as level sets. Hence $(\omega\mu)^{-1}(0)$ is diffeomorphic to $(\omega\mu)^{-1}(t)$. Thus M_t is diffeomorphic to $\nu(M_0, \mathscr{H}_0)$.

Remark. There is no need to introduce the function τ into this proof. We could have used the function r instead. However for later proofs we must use $e(q)$, so in analogy with these later proofs we introduced $e(q)$ into this proof. In fact $\nu(\mathscr{M}^+)$ is already a smooth manifold in this case.

3. The nonsingular model in Case II

According to our definition in § 0 a Case II space $(M, \mathscr{H})$ is at worse locally isomorphic to a complex model of the form $(A, \mathscr{A}_{e,f})$ where

$$A = \{z : z_1 z_2 = 0, z = (z_1, \cdots, z_n) \in D \subset \boldsymbol{C}^n\} , \qquad \mathscr{A}_{e,f} = \mathscr{O}/(z_1^e z_2^f) ,$$

D is an open neighborhood of $0 \in \boldsymbol{C}^n$, $\mathscr{O} = \mathscr{O}(D)$, and e, f are nonnegative integers. (We assume that *somewhere* $(M, \mathscr{H})$ is locally isomorphic to a model $(A, \mathscr{A}_{e,f})$ with $e > 0, f > 0$.) The integers e, f of course depend on the locality. As a set $M = \bigcup \overline{M}_j$, a finite union of connected nonsingular compact complex manifolds such that no point of M belongs to three or more $\overline{M}_j$; and to each $\overline{M}_j$ we associate integers e_j (the *multiplicity* of $\overline{M}_j$). These integers are defined as follows:

(i) If $p \in M, p$ in *exactly one* $\overline{M}_j$, then near $p, (M, \mathscr{H})$ is isomorphic to $(A, \mathscr{A}_{e_j,0})$.

(ii) If $p \in \overline{M}_j \cap \overline{M}_k$, then near $p, (M, \mathscr{H})$ is isomorphic to $(A, \mathscr{A}_{e_j,e_k})$. (So near $p, \overline{M}_j$ will be given by $z_1 = 0$, $\overline{M}_k$ by $z_2 = 0$.)

For the present let us consider the case $M = \overline{M}_1 \cup \overline{M}_2$, and let $e = e_1$, $f = e_2$. The case $M = \bigcup_{j=1}^{s} \overline{M}_j$, $s > 2$, is not essentially harder. Let

$$A_z = \{z \in D \subset C^n : z_1 z_2 = 0\}\ , \qquad A_s = \{\zeta \in \Delta \subset C^n : \zeta_1 \zeta_2 = 0\}\ ,$$

where $z = (z_1, \cdots, z_n)$, $\zeta = (\zeta_1, \cdots, \zeta_n)$, and D, Δ are open neighborhoods of 0. Let

$$_z\mathscr{A}_{e,f} = \mathscr{O}(D)/(z_1^e z_2^f)\ , \qquad _\zeta\mathscr{A}_{e,f} = \mathscr{O}(\Delta)/(\zeta_1^e \zeta_2^f)\ .$$

Suppose $\Phi = (\varphi, \varphi^*)$ is an isomorphism

$$\Phi : (A_{z}, {_z\mathscr{A}_{e,f}}) \to (A_{\zeta}, {_\zeta\mathscr{A}_{e,f}})\ .$$

We know (§ 0) that Φ is induced by a map $z \mapsto (F_1(z), \cdots, F_n(z))$.

Lemma 1. $F_1(z) = f(z)z_1$, $F_2(z) = g(z)z_2$, *where* $f(0, 0, z_3, \cdots, z_n) \neq 0$, $g(0, 0, z_3, \cdots, z_n) \neq 0$.

We omit the easy proof.

In what follows we shall assume the dimension of M is 2. The case $\dim M > 2$ is not essentially different. We divide the discussion into parts as follows.

Part (*i*). We assume that the greatest common divisor $(e, f) = 1$, and that both $e > 1$ and $f > 1$. Let $M = \{U_j\} \cup \{V_k\} \cup \{W_l\}$, a union of open subsets of M where $U_j \subset \overline{M}_1$, $V_k \subset \overline{M}_2$ and the sets $\{W_l\}$ cover $\overline{R} = \overline{M}_1 \cap \overline{M}_2$. We assume each $(U_j, \mathscr{H})$ is isomorphic to a model $(B, \mathscr{B}_e)$, $(V_k, \mathscr{H})$ to a model $(B, \mathscr{B}_f)$, and $(W_l, \mathscr{H})$ to $(A, \mathscr{A}_{e,f})$, where $B = \{z \in D \subset C^3 : z_1 = 0\}$ and D is an open disk around $0 \in C^3$. We define bundles E_1 and F_1 on $\overline{M}_1$ as follows. Let $W_l^1 = \overline{M}_1 \cap W_l$. Then $\{U_j\} \cup \{W_l^1\}$ is an open covering of $\overline{M}_1$. Let $z = (z_{1j}, z_{2j}, z_{3j})$ on U_j or W_j^1. We then have

$$(1) \qquad z_{1j} = e_{jk}(z_{1k}, z_{2k}, z_{3k}) z_{1k} \qquad \text{on } U_j \cap U_k \text{ (or on } U_j \cap W_k^1 \text{ or } W_j^1 \cap W_k^1)\ .$$

As in § 1, or by Lemma 1, we see that $e_{jk}(0, z_{2k}, z_{3k})$ is a nonvanishing holomorphic function, and we set

$$(2) \qquad e_{jk}(0, z_{2k}, z_{3k}) = e_{jk}(z_{2k}, z_{3k})\ .$$

The 1-cocycle $\{e_{jk}\}$ defines the line bundle E_1, and the divisor $\overline{R} \subset \overline{M}_1$ defines the line bundle F_1. Using the fact that $(M, \mathscr{H})$ can occur as a fibre in a one-parameter family one can easily verify that $F_1^f \cdot E_1^e$ is the trivial line bundle on $\overline{M}_1$. Notice that F_1 is defined by the 1-cocycle $f_{jk}(z_{2k}, z_{3k})$ where $\overline{R} = \{z_{2k} = 0\}$ and

$$(3) \qquad z_{2j} = f_{jk}(z_{2k}, z_{3k}) z_{2k}\ .$$

We proceed in a similar manner to define line bundles E_2 and F_2 on M_2, and we find that $E_2^e F_2^f$ is the trivial line bundle. Notice that E_2 is the bundle of the divisor $\bar{R} \subset M_2$.

We now begin the discription of the nonsingular model $\nu(M, \mathscr{H})$. Since $E_1^e F_1^f$ is trivial, by choosing our covering appropriately we can find nonvanishing holomorphic functions u_j such that

$$(4) \qquad f_{jk}^f e_{jk}^e = u_j / u_k \ .$$

Now consider the line bundle E_1^{-1} on $\bar{M}_1$. Let ξ_j be a fibre coordinate for E_1^{-1} over W_j^1 (or U_j). Then one can easily check that the equations

$$(5) \qquad u_j \xi_j^e = z_{2j}^f \quad \text{on } W_j^1 \ , \qquad u_j \xi_j^e = 1 \quad \text{on } U_j$$

define local varieties which fit together to give a global subvariety V_1 of E_1^{-1}. Let $\pi: E_1^{-1} \to \bar{M}_1$ be the projection map. Then we see that π makes V_1 an l-sheeted covering of $\bar{M}_1$ branched over $\bar{R}$. Notice that if $f_{jk}^f e_{jk}^e = u_j' / u_k'$ for some other set of nonvanishing holomorphic functions, then $u_j' = C u_j$ where C is a nonzero constant (independent of j of course). We want our constructions to be independent of the particular trivialization $\{u_j\}$. Notice that $C u_j \xi_j^e = z_{2j}^f$ can be mapped onto $u_j \xi_j^e = z_{2j}^f$ by sending (ξ_j, z_{2j}, z_{3j}) to $(C^{1/e}\xi_j, z_{2j}, z_{3j})$, and this map defines an isomorphism. So V_1 is well defined independent of the particular choice of the $\{u_j\}$. We let $R_1 = \pi^{-1}(\bar{R})$.

We define a differentiable manifold J_1 with boundary as follows. Let $S^1 = \{z \in C^n : |z| = 1\}$. We think of S^1 as a multiplicative group. We suppose that $W_j^1 = \{(z_{2j}, z_{3j}) : |z_{j2}| < 1, |z_{j3}| < 1\}$ and we introduce spaces

$$\hat{W}_j^1 = \{(r_j, \theta_j, z_{3j}) : 0 \leq r_j < 1, \theta_j \in S^1, |z_{3j}| < 1\} \ ,$$

choose v_j so that

$$(6) \qquad v_j^e = u_j \ .$$

Then let

$$e_{jk}(z_{2k}, z_{3k}) = |e_{jk}(z_{2k}, z_{3k})| \tau_{jk}(z_{2k}, z_{3k}) \ ,$$
$$f_{jk}(z_{2k}, z_{3k}) = |f_{jk}(z_{2k}, z_{3k})| \sigma_{jk}(z_{2k}, z_{3k}) \ ,$$
$$v_k(z_{2k}, z_{3k}) = |v_k(z_{2k}, z_{3k})| \beta_k(z_{2k}, z_{3k}) \ ,$$

where $\tau_{jk}, \tau_{jk}, \beta_k$ take values in S^1. Notice that we have maps

$$(7) \qquad \lambda: (r_j, \theta_j, z_{3j}) \to (r_j^f \theta_j^f / v_j(r_j^e \theta_j^e, z_{3j}), r_j^e \theta_j^e, z_{3j})$$

on $\hat{W}_j^1$, where (ξ_j, z_{2j}, z_{3j}) are coordinates on E_1^{-1}. This map λ is an isomorphism of $\{(r_j, \theta_j, z_{3j}) \in \hat{W}_j^1 : r_j > 0\}$ onto $\pi^{-1}(W_j^1 - \bar{R}) \cap V_1$. We then form a union $\hat{W}^1 = \cup \hat{W}_j^1$ by identifying $(r_j, \theta_j, z_{3j}) \in \hat{W}_j^1$ with $(r_k, \theta_k, z_{3k}) \in \hat{W}_k^1$ if and only if

$$(8)\qquad \begin{aligned} r_j^e &= |f_{jk}(r_k^e\theta_k^e, z_{3k})| r_k^e\ , & \theta_j^e &= \sigma_{jk}(r_k^e\theta_k^e, z_{3k})\theta_k^e\ , \\ \theta_j^f &= \left[\frac{\tau_{jk}^{-1}(r_k^e\theta_k^e, z_{3k})\beta_j(r_k^e\theta_k^e, z_{3k})}{\beta_k(r_k^e\theta_k^e, z_{3k})}\right]\theta_k^f\ , & z_{3j} &= h_{jk}(r_k^e\theta_k^e, z_{3k})\ , \end{aligned}$$

where we may think of β_j as a holomorphic function of $(z_{2k}, z_{3k}) \in W_j^1 \cap W_k^1$. That the second two equations uniquely define θ_j if (r_k, θ_k, z_{3k}) are given follows from the following lemma and the fact that

$$(9)\qquad \sigma_{jk}^f \tau_{jk}^e = \beta_j^e / \beta_k^e\ .$$

Lemma 2. *Let $a, b, c, d \in S^1 = \{z : |z| = 1\}$. Suppose e, f are relatively prime integers, greater than 1. Consider the following equations for a*

$$(10)\qquad a^e = cb^e\ , \qquad a^f = db^f\ .$$

If b, c, d are fixed with $c^f = d^e$, then a is uniquely determined, i.e., there is a unique solution to these equations.

We omit the easy proof.

We thus get a manifold $\hat{W}^1$ with boundary and a map $\lambda : \hat{W}^1 \to V_1$ such that $\lambda(\partial\hat{W}^1) = R_1$, and λ maps $\hat{W}^1 - \partial\hat{W}^1$ isomorphically onto $\pi^{-1}(\bigcup W_j^1 - \bar{R}) \cap V_1$. We can thus form

$$\begin{aligned} J_1 &= (V_1 - R_1) \cup \partial\hat{W}^1 \qquad \text{(disjoint union)} \\ &= (V_1 - R_1) \cup \hat{W}^1 \end{aligned}$$

with the identification made by λ. J_1 is a manifold with boundary $\partial J_1 = \partial\hat{W}^1$. This boundary is an S^1 bundle over $R_1 = \bar{R}$, which can be described as follows:

$$\partial J_1 = \bigcup \{(\theta_j, z_{3j}) : |z_{3j}| < 1, \theta_j \in S^1\}\ ,$$

where (θ_j, z_{3j}) is identified with (θ_k, z_{3k}) if and only if

$$(11)\qquad \begin{aligned} &\theta_j^e = \sigma_{jk}(0, z_{3k})_k^e\ , \qquad \theta_j^f = \left[\frac{\tau_{jk}^{-1}(0, z_{3k})\beta_j(0, z_{3k})}{\beta_k(0, z_{3k})}\right]\theta_k^f\ , \\ &z_{3j} = h_{jk}(0, z_{3k}) = h_{jk}(z_{3k})\ . \end{aligned}$$

We define V_2, J_2 similarly for $\bar{M}_2$. V_2 is an f-sheeted covering of $\bar{M}_2$ branched over $\bar{R}$, and $J_2 = (V_2 - R_2) \cup \partial J_2$. ∂J_2 is an S^1 bundle over $\bar{R}$ such that

$$\partial J_2 = \bigcup \{(\varphi_j, z_{3j}) : |z_{3j}| < 1, \varphi_j \in S^1\}\ ,$$

where (φ_j, z_{3j}) is identified with (φ_k, z_{3k}) if and only if

$$(12)\qquad \begin{aligned} &\varphi_j^e = \left[\frac{\sigma_{jk}^{-1}(0, z_{3k})\alpha_j(0, z_{3k})}{\alpha_k(0, z_{3k})}\right]\varphi_k^e\ , \\ &\varphi_j^f = \tau_{jk}(0, z_{3k})\varphi_k^f\ , \qquad z_{3j} = h_{jk}(z_{3k})\ , \end{aligned}$$

where $\alpha_k^f(0, z_{3k}) = \beta_k^e(0, z_{3k})$. Now ∂J_1 and ∂J_2 are diffeomorphic. In fact, by using

$$\sigma_{jk}^f \tau_{jk}^e = \beta_j^e / \beta_k^e = \alpha_j^f / \alpha_k^f \ , \tag{13}$$

Lemma 2, and equations (11) and (12) one can easily check that the equations

$$\varphi_j^f = \beta_j(0, z_{3j})\theta_j^{-f} \ , \quad \varphi_j^e = \alpha_j(0, z_{3j})\theta_j^{-e} \ , \quad z_{3j} = z_{3j} \tag{14}$$

give a well defined diffeomorphism of ∂J_1 with ∂J_2, and thus we form $J_1 \cup J_2$ where we identify ∂J_1 and ∂J_2 to get a topological 4-manifold (without boundary). Then $J_1 \cup J_2 = \nu(M, \mathscr{H})$ is the nonsingular model for $(M, \mathscr{H})$. Since the trivializations of $F_1^f \cdot E_1^e, F_2^f \cdot E_2^e$ are uniquely determined up to a constant factor so are the functions β_j, α_j. Thus as a topological manifold $\nu(M, \mathscr{H})$ is uniquely determined.

Part (*ii*). We assume that $e > 1, f > 1$, and that $e = al, f = bl$ with the greatest common divisor $(a, b) = 1$. We define the variety V_1 in the same way. However instead of $\hat{W}_j^1$ we have $\bigcup_{p=0}^{l-1} \hat{W}_{jp}^1$ (disjoint union), where each $\hat{W}_{jp}^1 = \{(r_j, \theta_j, z_{3j})\colon 0 \leq r_j < 1, \theta_j \in S^1, |z_{3j}| < 1\}$. The variety V_1 is locally given by equations $(v_j \xi_j)^{al} = z_{2j}{}^{bl}$, where $v_j{}^{al} = u_j$, $f_{jk}{}^{bl} e_{jk}{}^{al} = u_j / u_k$, as before. In this case the map $\lambda\colon \hat{W}_{jp}^1 \to V_1$ is given by

$$\lambda(r_j, \theta_j, z_{3j}) = (\eta^p r_j^b \theta_j^b / v_j(r_j^a \theta_j^a, z_{3j}), r_j^a \theta_j^a, z_{3j}) \tag{15}$$

for $(r_j, \theta_j, z_{3j}) \in \hat{W}_{jp}^1$, where $\eta = \exp(2\pi i / al)$. With the notation as before we identify $(r_j, \theta_j, z_{3j}) \in \hat{W}_{jp}^1$ with $(r_k, \theta_k, z_{3k}) \in \hat{W}_{kp}^1$, (where $k \neq j$) if and only if

$$\begin{aligned} r_j^a &= |f_{jk}(r_k^a \theta_k^a, z_{3k})| r_k^a \ , & \theta_j^a &= \sigma_{jk}(r_k^a \theta_k^a, z_{3k})\theta_k^a \ , \\ \eta^p \theta_j^b &= \frac{\beta_j(r_k^a \theta_k^a, z_{3k})}{\beta_k(r_k^a \theta_k^a, z_{3k})} \tau_{jk}^{-1}(r_k^a \theta_k^a, z_{3k}) \eta^q \theta_k^b \ , & z_{3j} &= h_{jk}(r_k^a \theta_k^a, z_{3k}) \ . \end{aligned} \tag{16}$$

To see that these equations uniquely determine an identification we need the following lemma.

Lemma 3. *Let $x, y, \mu, \nu \in S^1$, and let $\eta = \exp(2\pi i / al)$, where $(a, b) = 1$, and a, b, l are integers greater than 1. Consider the following equations*:

$$\eta^p x^b = \mu \eta^q y^b \ , \qquad x^a = \nu y^a \ . \tag{17}$$

If y, μ, ν, and q are fixed with $y \in S^1$, $\mu^{al} = \nu^{bl}$, and $0 \leq q < l$, then there are a unique $x \in S^1$ and p with $0 \leq p < l$ satisfying (17).

We omit the easy proof.

Again we get a manifold $\hat{W}^1 = \bigcup \hat{W}_{jp}^1$ with boundary $\partial \hat{W}^1$. In this case $\partial \hat{W}^1$ is a fibre space over $\bar{R}$ with fibre a union of l circles. We now construct

$$\begin{aligned} J_1 &= (V_1 - R_1) \cup \partial \hat{W}^1 \qquad \text{(disjoint union)} \\ &= (V_1 - R_1) \cup \hat{W}^1 \ , \end{aligned}$$

where the identification is made by λ as before. We construct J_2 similarly, and form $J_1 \cup J_2 = \nu(M, \mathscr{H})$.

Part (iii). We assume at least one of the integers is 1. Suppose $e = 1, f > 1$. Then $V_1 = \overline{M}_1$, and V_2 is described as before. The description of J_1 is easy, and J_2 is as before. Again one checks that $J_1 \cup J_2$ gives a manifold which we call $\nu(M, \mathscr{H})$. Notice we cannot define E_1 in this case, but it is not needed since we take $V_1 = \overline{M}_1$. If $e = f = 1$, we take $V_1 = \overline{M}_1$, $V_2 = \overline{M}_2$. The reader should be able to supply the rest of the details. This ends the description of $\nu(M, \mathscr{H})$.

4. Examples

First we make a remark which is a consequence of the discussion in § 3. We use the notation of § 4, so $M = \overline{M}_1 \cup \overline{M}_2$ and $\mathscr{H}$ is also the same. The following proposition is then a consequence of the discussion there.

Proposition. *Suppose that $(M, \mathscr{H})$ occurs as a fibre in a one-parameter family, and further that $(M, \mathscr{H})$ is a Case* II *space. Let $\bar{R} = \overline{M}_1 \cap \overline{M}_2$, and let η_i be the (holomorphic) normal bundle of $\bar{R}$ in $\overline{M}_i$. Let the integers e and f be defined as in* § 3. *Then $\eta_1^f \eta_2^e = 1$ (holomorphically trivial).*

Remark. This gives a necessary condition for a space $(M, \mathscr{H})$ to occur as a fibre in a one-parameter family.

Example 1. Let $M = \{(\zeta_0, \zeta_1, \zeta_2, \zeta_3) \in \boldsymbol{P}^3 \mid \zeta_0 = 0 \text{ or } \zeta_1 = 0\}$. Then $M = W_1 \cup W_2$, and each W_i is isomorphic to $\boldsymbol{P}^2$. The intersection $W_1 \cap W_2$ is isomorphic to $\boldsymbol{P}^1$. Let $\mathcal{O}$ be the sheaf of germs of holomorphic functions on $\boldsymbol{P}^3$, and $\mathscr{I}$ be the ideal sheaf of germs of holomorphic functions on $\boldsymbol{P}^3$ which vanish on M. Let $\mathscr{H} = (\mathcal{O}/\mathscr{I})|_M$. Then we claim $(M, \mathscr{H})$ *cannot* occur as a fibre in a one-parameter family. For, let N_i be the bundle of the divisor $W_1 \cap W_2$ in W_i restricted to $W_1 \cap W_2$. Then $N_1 \cdot N_2 = [2p]$ where p is a point of $W_1 \cap W_2 = \boldsymbol{P}^1$. Since $[2p]$ is not trivial, the proposition implies that $(M, \mathscr{H})$ cannot occur as a fibre in a one-parameter family. If we allow $\mathscr{H}$ to have nilpotents, say let W_i have multiplicity e_i, then we get $N_1^{e_1} \cdot N_2^{e_2} = [(e_1 + e_2)p]$. Thus we see that the underlying space M has no structure sheaf $\mathscr{J}$ such that $(M, \mathscr{J})$ occurs as a fibre in some one-parameter family.

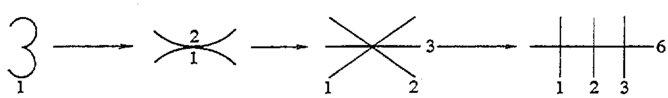
Example 2. We recall some definitions from Kodaira's paper [5]. By an *elliptic surface* we mean a triple (V, Φ, R) where V is a connected complex compact manifold of complex dimension 2, R is a nonsingular algebraic curve (compact Riemann surface), Φ is a proper surjective holomorphic map, and the general fibre $\Phi^{-1}(u)$ is a nonsingular elliptic curve. Thus an elliptic surface is a one-parameter family of complex spaces of dimension 1. Assuming that all of the fibres are free from nonsingular rational curves C with self intersection $(C^2) = -1$, Kodaira has given a list of every possible singular fibre which can occur in an elliptic surface (see Kodaira [5, Theorem 6.2]). We shall verify that for each of these singular spaces $(M, \mathscr{H})$ that $\nu(M, \mathscr{H})$ is a torus thus checking the theorem of § 5 in these cases.

(i) The first type of singular fibre listed by Kodaira is a nonsingular elliptic curve Θ with multiplicity $m > 1$. An m-sheeted unramified covering of Θ is again a torus so $\nu(\Theta, \mathscr{H})$ is a torus.

(ii) Next we consider a rational curve Θ with an ordinary double point with multiplicity $m > 0$. Θ is not yet a space with normal crossings. However, if we blow up the double point, we get a curve $m\Theta_1 + 2m\Theta_2$, where Θ_i are nonsingular rational curves, the integers preceeding the curves represent their multiplicities, and Θ_1, Θ_2 intersect normally in two points. Θ_1 is the proper transform of Θ. The g.c.d. $(m, 2m) = m$, so the boundaries of the varieties J_1 and J_2 (described in § 3) are bundles over two points, each with fibre a union of m circles. The varieties L_1 and L_2 from which J_1 and J_2 are constructed can be described as follows. L_1 is just a union of m copies of Θ_1 in which all m copies are pinched together at two distinct points. L_2 is formed from a union of m copies of a 2-sheeted covering of Θ_2 branched over two points with branching order 1 at each point. These branched coverings are all pinched together at two points. All of the points above one of the branch points are pinched to a single point, and all of the points above the other branch point are pinched to a single point. Thus $J_1 = \bigcup_{i=1}^m J_{1i}$, where each J_{1i} is a 2-sphere with two open disks removed. By using the Riemann-Hurwitz formula we see that $J_2 = \bigcup_{i=1}^m J_{2i}$, where each J_{2i} is a 2-sphere with two open disks removed. J_1 and J_2 are pasted together according to the following scheme. We glue J_{1i} to J_{2i} along one of the boundary circles, and J_{1i} is glued to J_{2i+1} along the other boundary circle where $i + 1$ is reduced modulo m. The resulting manifold is a closed chain of spheres glued together, and is clearly a torus.

(iii) Another possibility is $M = m\Theta_0 + m\Theta_1$ with $m > 1$, where the Θ_i are nonsingular rational curves, m is the multiplicity of each curve, and Θ_0 intersects Θ_1 normally in two distinct points. Thus $(M, \mathscr{H})$ is a space with normal crossings. To see that $\nu(M, \mathscr{H})$ is a torus is quite similar to (but easier than) the last part of the discussion in (ii).

(iv) $M = \Theta$ where Θ is a rational curve with one cusp. The multiplicity of M is one, so $\mathscr{H}$ is the reduced structure sheaf, i.e., $\mathscr{H}$ is the structure sheaf induced on Θ_0 considered as a subset of $\boldsymbol{P}^2$. A neighborhood of the cusp of the curve Θ is isomorphic to a neighborhood of the origin of the analytic set $\{(x, y) \in \boldsymbol{C}^2 \mid x^2 = y^3\}$. Thus $(M, \mathscr{H})$ is *not* a space with normal crossings. We perform a sequence of quadratic transformations to resolve the singularity of Θ. We represent this resolution by the following sequence of symbols:

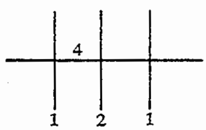


These symbols have the following meaning. The first symbol represents a rational curve with a cusp; it has multiplicity 1. We blow up the cuspidal point to get two nonsingular rational curves which are tangent at one point. The cuspidal point is replaced by a rational curve with multiplicity 2. The curve with the cusp has as proper transform a nonsingular rational curve with multiplicity 1. Next we blow up the point of tangency to get three nonsingular rational curves intersecting non-tangentially at the same point. The curve with multiplicity three is the result of blowing up the point of tangency. Finally we blow up the point of intersection to get a nonsingular rational curve of multiplicity 6 intersecting the three other rational curves normally (at three different points). This space is a space with normal crossings $(C, \mathscr{C})$ where $C = C_1 \cup C_2 \cup C_3 \cup C_6$. The curves C_k have multiplicity k. We must construct the branched coverings L_k of C_k described in § 3. L_6 is a 6-sheeted covering of $C_6 = \boldsymbol{P}^1$ branched over 3 points. Over the first point there is one branch point of order 6, over the next point there are two branch points of order 3, and over the last point there are three branch points of order 2. First we separate the branch points of order 3 (they are pinched together in L_6), and then we separate those of order 2. This gives a nonsingular ramified covering L_6' of C_6. Recall the Riemann-Hurwitz formula

$$g = \tfrac{1}{2} \sum_{\rho} (e_\rho - 1) - n + 1$$

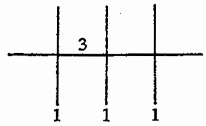
for a branched covering R of the sphere $\boldsymbol{P}^1$ where g is the genus of R, where the sum is over all branch points ρ in R, e is the branching order at ρ, and n is the number of sheets in the covering. With $R = L_6'$ we get $g(L_6') = 1$, and thus L_6' is a torus. Hence J_6 is a torus with six disks cut out of it. J_1 is clearly a sphere with a disk cut out. J_2 is a union of two disjoint spheres, each with a disk cut out. Finally J_3 is a union of three disjoint spheres each with a disk cut out (L_3 is a union of three spheres $\boldsymbol{P}^1$, pinched together at one point). Thus $J_6 + J_1 + J_2 + J_3 = \nu(\Theta, \mathscr{H})$ is a torus.

(v) $M = \Theta_1 + \Theta_2$ where Θ_1 and Θ_2 are nonsingular rational curves intersecting tangentially at a point p. One checks that resolving the singularities gives the following graph of nonsingular rational curves:



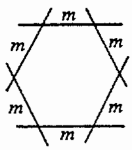
It is an easy exercise to verify that the nonsingular model is a torus.

(vi) $M = \Theta_1 + \Theta_2 + \Theta_3$ where Θ_1 are nonsingular rational curves intersecting transversally at a common point p. The structure sheaf is the reduced structure sheaf. The resolution of singularities gives:



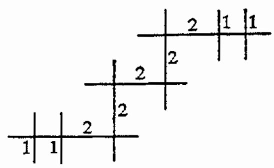
One easily verifies that the nonsingular model is a torus.

(vii) $M = m\Theta_1 + m\Theta_2 + \cdots + m\Theta_b$ where Θ_i is a nonsingular rational curve, m is its multiplicity, and the Θ_i are connected in a cyclic graph as follows:



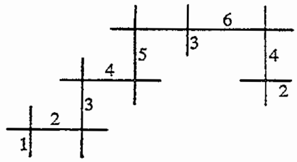
With the usual notation $J_k = J_{k1} \cup \cdots \cup J_{km}$ where each J_{kl} is a 2-sphere with two open disks cut out. Gluing these J_{kl} together cyclically one gets the nonsingular model which is clearly a torus.

(viii) The next possibility is $M = \Theta_1 + \cdots + \Theta_4 + 2\Theta_5 + 2\Theta_6 + \cdots + 2\Theta_b$, $b \geq 5$. The nonsingular rational curves Θ_j are connected together in the following graph:



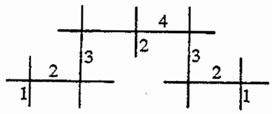
In this picture $b = 9$. All of the intersections are normal, and the multiplicities of the curves are given by the adjacent integers. The curves are labeled as follows. The four outside curves are Θ_i, $1 \leq i \leq 4$. Θ_5 intersects $\Theta_1, \Theta_2, \Theta_6$. Θ_6 intersects Θ_5 and Θ_7. This pattern continues until we get to Θ_b which intersects Θ_{b-1}, Θ_3, and Θ_4. The space J_k, $1 \leq k \leq 4$, is a 2-sphere with one open disk removed. The space J_k, $5 < k < b$, is a union of two 2-spheres, each 2-sphere with two open disks removed. Using the Riemann-Hurwitz formula one finds that J_5 and J_b are 2-spheres each with three disks cut out. One easily sees that the nonsingular model $\bigcup_{k=1}^{b} J_k$ is a torus.

(ix) $M = \Theta_1 + 2\Theta_2 + 3\Theta_3 + 4\Theta_4 + 5\Theta_5 + 6\Theta_6 + 4\Theta_7 + 3\Theta_8 + 2\Theta_9$. These nine nonsingular rational curves are connected together in the following graph:



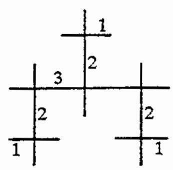
From the graph one easily sees which curves Θ_j intersect. Using the Riemann-Hurwitz formula one easily finds that J_k is a 2-sphere with two disks cut out for $2 \leq k \leq 5$. J_1 is a 2-sphere with one disk cut out. J_8 is a union of three 2-spheres, each with a disk cut out. J_9 is a union of two 2-spheres, each with a disk cut out. J_7 is a union of two 2-spheres, each with two disks cut out. J_6 is a torus with six disks cut out. The union is clearly a torus.

(x) $M = \Theta_1 + 2\Theta_2 + 3\Theta_3 + 4\Theta_4 + 3\Theta_5 + 2\Theta_6 + 2\Theta_7 + \Theta_8$. Each Θ_j is a nonsingular rational curve. They are joined in the following graph:



The notation is such that Θ_6 intersects Θ_4. Again apply the Riemann-Hurwitz formula to find that the nonsingular model is a torus.

(xi) The final example has the following graph (each curve is nonsingular rational):

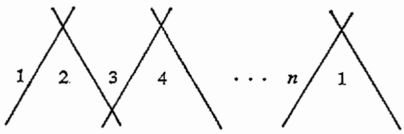


Follow the standard procedure, and find that the nonsingular model is a torus.

(xii) These examples yield the following information. If C is one of the singular fibres of an elliptic surface, and M is an arbitrary compact complex manifold, then $\nu(C \times M, \mathscr{H}) = T^2 \times M$, where $\mathscr{H}$ is chosen appropriately, and T^2 is a 2-torus. For example, let C be the curve discussed in Example 2 (xi). Then $\mathscr{H}$ is the structure sheaf which can be defined on $C \times M$ in an obvious way locally, and these local pieces can thus be fitted together to give $\mathscr{H}$.

Example 3. Let $\mathscr{N} = \boldsymbol{P}^1 \times \Delta$, $\Delta = \{z : z \in \boldsymbol{C}, |z| < \varepsilon\}$. Blow up a point on $\boldsymbol{P}^1 \times \{0\}$ to get a curve $C = \Gamma_1 \cup \Gamma_2$, where each $\Gamma_i \cong \boldsymbol{P}^1$, and Γ_1 intersects Γ_2 normally at one point. We get a family $\mathscr{M} \xrightarrow{\Pi} \Delta$ with $\Pi^{-1}(0) = C$ and $\Pi^{-1}(t) = \boldsymbol{P}$, $t \neq 0$. The naturally induced structure sheaf $\mathscr{H}$ on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is the reduced structure sheaf, and one easily computes that $\nu(C, \mathscr{H}) = S^2$.

Thus the theorem is verified in this case. Notice that the only complex structure on S^2 is that of $\boldsymbol{P}^1$. We could go further with this idea. One could produce for example a one-parameter family $\mathscr{L} \xrightarrow{\Pi} \Delta$, with $\Pi^{-1}(t) = \boldsymbol{P}^1$ if $t \neq 0$ and $\Pi^{-1}(0)$ being a collection of nonsingular rational curves represented by the following graph, where each intersection is normal, and the integers represent the multiplicity of Π on each curve:



One easily sees that $\nu(\Pi^{-1}(0), \mathscr{H}) = S^2$ where $\mathscr{H}$ is the structure sheaf. It is obvious that one could easily produce examples with $\nu(C, \mathscr{H}) = S^2$ whose graphs are much more complicated. (The author would like to thank the referee for the idea of this example.)

Example 4. Let $\boldsymbol{P}^1 = W_1 \cup W_2$, where $W_i = \boldsymbol{C}$, and the coordinates z_i on W_i are related by the equation $z_1 z_2 = 1$. Let $R^{(m)} = \boldsymbol{P}^1 \times W_1 \cup \boldsymbol{P}^1 \times W_2$ where (w_1, z_1) is identified with (w_2, z_2) if and only if $z_1 z_2 = 1$ and $w_2 = z_1^m w_1$. This is a $\boldsymbol{P}^1$ bundle over $\boldsymbol{P}^1$ (a ruled surface). Then $S^{(m)}$ is a surface with an ordinary double curve (a Case II surface with $e = m = 1$) obtained from $R^{(m)}$ by identifying the two sections D_0 and D_∞, which are defined respectively by $w_1 = w_2 = 0$ and $w_1 = w_2 = \infty$. If we let $\mathscr{H}^{(m)}$ be the reduced structure sheaf on $S^{(m)}$, then Kodaira has shown [6] that $\nu(S^{(m)}, \mathscr{H}^{(m)}) = S^1 \times S^3$ for $m \geq 1$. If $m = 0$, then $S^{(m)}$ is $\mathscr{R} \times \boldsymbol{P}^1$ where $\mathscr{R}$ is a rational curve with a double point. Then by Example 2 (ii) and 2 (xii), the nonsingular model is $T^2 \times S^2$.

5. The deformation theorem for Case II spaces

Theorem. *Let $(\mathscr{M}, \varpi, \Delta) = \{M_t : t \in \Delta\}$ be a one-parameter family. Suppose M_t is nonsingular for $t \neq 0$, and $(M_0, \mathscr{H}_0)$ is a Case* II *space. Then M_t is homeomorphic to $\nu(M_0, \mathscr{H}_0)$ for $t \neq 0$.*

(The proof of this theorem should be divided into parts as in the discussion in § 3. However, we shall only supply the proof corresponding to part (i) of the discussion in § 3. The proofs for the other parts are similar. For simplicity we assume $\dim \mathscr{M} = 3$.)

Proof. Remember we are assuming that $\Delta = \{z \in \boldsymbol{C} : |z| < 1\}$. We shall suppose that $M_0 = X \cup Y$ (as in § 3) and that we have a covering $\{\mathscr{W}_j\} \cup \{\mathscr{U}_j\} \cup \{\mathscr{V}_j\}$ of M_0 by open sets in $\mathscr{M}$ with coordinates (z_{1j}, z_{2j}, z_{3j}) such that

$$\varpi(z_{1j}, z_{2j}, z_{3j}) = z_{1j}^e z_{2j}^f \qquad \text{on } \mathscr{W}_j ,$$

$$\varpi(z_{1j}, z_{2j}, z_{3j}) = z_{1j}^e \qquad \text{on } \mathscr{U}_j ,$$

$$\varpi(z_{1j}, z_{2j}, z_{3j}) = z_{2j}^f \qquad \text{on } \mathscr{V}_j .$$

We suppose that X is given by $\{z_{1j} = 0\}$, and Y is defined by $\{z_{2j} = 0\}$. On $\mathscr{W}_j \cap \mathscr{W}_k$ we have

$$(1) \qquad \begin{aligned} z_{1j} &= e_{jk}(z_{1k}, z_{2k}, z_{3k}) \cdot z_{1k} , \\ z_{2j} &= f_{jk}(z_{1k}, z_{2k}, z_{3k}) \cdot z_{2k} , \\ z_{3j} &= h_{jk}(z_{1k}, z_{2k}, z_{3k}) . \end{aligned}$$

On $\mathscr{W}_k \cap \mathscr{U}_j$ we have

$$(2) \qquad \begin{aligned} z_{1j} &= e_{jk}(z_{1k}, z_{2k}, z_{3k}) \cdot z_{1k} , \\ z_{2j} &= f_{jk}(z_{1k}, z_{2k}, z_{3k}) , \\ z_{3j} &= h_{jk}(z_{1k}, z_{2k}, z_{3k}) . \end{aligned}$$

Similarly we have on $\mathscr{W}_k \cap \mathscr{V}_j$

$$(3) \qquad \begin{aligned} z_{1j} &= e_{jk}(z_{1k}, z_{2k}, z_{3k}) , \\ z_{2j} &= f_{jk}(z_{1k}, z_{2k}, z_{3k}) \cdot z_{2k} , \\ z_{3j} &= h_{jk}(z_{1k}, z_{2k}, z_{3k}) . \end{aligned}$$

We shall assume that $\mathscr{U}_j \cap \mathscr{V}_k = \phi$ for all j, k.

Let $\mathscr{W} = \bigcup \mathscr{W}_j$. We define continuous functions r and s on $\mathscr{W}$ as follows. We let $\{a_j\}$ be a system of positive differentiable functions on $\mathscr{W}_j$ such that

$$(4) \qquad |e_{jk}| = a_k / a_j \qquad \text{on } \mathscr{W}_j \cap \mathscr{W}_k .$$

Then

$$(5) \qquad |f_{jk}| = b_k / b_j ,$$

where

$$(6) \qquad b_k = a_k^{-e/f} .$$

We let

$$x_j = a_j z_{1j} , \quad y_j = b_j z_{2j} , \quad z_j = z_{3j} ,$$

and introduce differentiable coordinates (x_j, y_j, z_j) on $\mathscr{W}_j$. We may assume

$$\mathscr{W}_j = \{(x_j, y_j, z_j) \colon |x_j| < \varepsilon, |y_j| < \varepsilon, |z_j| < 1\} ,$$

where $\varepsilon > 0$ is a small number. Since $|x_j| = |x_k|$ and $|y_j| = |y_k|$ on $\mathscr{W}_j \cap \mathscr{W}_k$, $r = |x_j| = |x_k|$ and $s = |y_j| = |y_k|$ define nonnegative continuous functions on $\mathscr{W}$. On $\mathscr{W}_j$ the function ϖ is given by $\varpi(x_j, y_j, z_j) = x_j^e y_j^f$.

If $\Delta^+ = \{t \colon 0 \leq t < 1\}$, then we set $\mathscr{M}^+ = \varpi^{-1}(\Delta^+)$. From now on we will restrict our attention to this subspace. Let $\mathscr{W}_j^+ = \mathscr{M}^+ \cap \mathscr{W}_j$ so that

$$\mathscr{W}_j^+ = \{(x_j, y_j, z_j) \colon x_j^e y_j^f \geq 0, |x_j| < \varepsilon, |y_j| < \varepsilon, |z_j| < 1\} .$$

We set

$$\nu(\mathscr{W}_j^+) = \{(r, s, \theta_j, z_j) \colon 0 \leq r < \varepsilon, 0 \leq s < \varepsilon, \theta_j \in S^1, |z_j| < 1\} .$$

Consider equations (1) and think of the functions e_{jk}, , f_{jk}, and h_{jk} as functions of (x_k, y_k, z_k). Then

$$(7) \qquad \begin{aligned} e_{jk}(x_k, y_k, z_k) &= |e_{jk}(x_k, y_k, z_k)|\tau_{jk}(x_k, y_k, z_k) , \\ f_{jk}(x_k, y_k, z_k) &= |f_{jk}(x_k, y_k, z_k)|\sigma_{jk}(x_k, y_k, z_k) . \end{aligned}$$

Then we construct a space

$$\nu(\mathscr{W}^+) = \bigcup_j \nu(\mathscr{W}_j^+) ,$$

by identifying $(r, s, \theta_j, z_j) \in \nu(\mathscr{W}_j^+)$ with $(r, s, \theta_k, z_k) \in \nu(\mathscr{W}_k^+)$ if and only if

$$(8) \qquad \begin{aligned} \theta_j^f &= \tau_{jk}^{-1}(r\theta_k^{-f}, s\theta_k^e, z_k)\theta_k^f , \\ \theta_j^e &= \sigma_{jk}(r\theta_k^{-f}, s\theta_k^e, z_k)\theta_k^e , \\ z_j &= h_{jk}(z_{1k}, z_{2k}, z_{3k}) . \end{aligned}$$

As before these equations uniquely determine an identification, so $\nu(\mathscr{W}^+)$ is well defined.

On $\mathscr{U}_j \cap \mathscr{U}_k$ we have

$$(9) \qquad \begin{aligned} z_{1j} &= e_{jk}(z_{1k}, z_{2k}, z_{3k})z_{1k} , \\ z_{2j} &= f_{jk}(z_{1k}, z_{2k}, z_{3k}) , \\ z_{3j} &= h_{jk}(z_{1k}, z_{2k}, z_{3k}) . \end{aligned}$$

We let

$$\begin{aligned} \nu(\mathscr{U}_j^+) = \{(R, \theta_j, z_{2j}, z_{3j}) \colon 0 \leq R \leq \varepsilon, \theta_j \in S^1, \\ \theta_j^e = 1, |z_{2j}| < 1, |z_{3j}| < 1\} . \end{aligned}$$

Notice that $e_{jk}^e = 1$ since $z_{1j}^e = z_{1k}^e$. We construct a space $\nu(\mathscr{U}^+) = \bigcup_j \nu(\mathscr{U}_j^+)$, where $(R, \theta_j, z_{2j}, z_{3j}) \in \nu(\mathscr{U}_j^+)$ is identified with $(R, \theta_k, z_{2k}, z_{3k}) \in \nu(\mathscr{U}_k^+)$ if and only if

$$(10) \qquad \begin{aligned} \theta_j &= e_{jk}^{-1}(R\theta_k^{-1}, z_{2k}, z_{3k})\theta_k , \\ z_{2j} &= f_{jk}(R\theta_k^{-1}, z_{2k}, z_{3k}) , \\ z_{3j} &= h_{jk}(R\theta_k^{-1}, z_{2k}, z_{3k}) . \end{aligned}$$

Notice that $R = |z_{1j}| = |z_{1k}|$ is a well defined function on $\bigcup \mathscr{U}_j = \mathscr{U}$. We have a similar construction for $\nu(\mathscr{V}^+) = \bigcup_j \nu(\mathscr{V}_j^+)$.

Next we want to construct $\nu(\mathscr{M}^+)$. First we define $\nu(\mathscr{V}^+) \cup \nu(\mathscr{W}^+) \cup \nu(\mathscr{U}^+)$ where, for example we identify $(r, s, \theta_k, z_k) \in \nu(\nu\mathscr{W}_k^+)$ with $(R, \theta_j, z_{2j}, z_{3j}) \in \nu(\mathscr{U}_j^+)$ if and only if

$$(11) \qquad \begin{aligned} R &= s^{f/e}r \;, & \theta_j &= \tau_{jk}^{-1}(r\theta_k^{-f}, s\theta_k^e, z_k)\theta_k^f \;, \\ z_{2j} &= f_{jk}(r\theta_k^{-f}, s\theta_k^e, z_k) \;, & z_{3j} &= h_{jk}(r\theta_k^{-f}, s\theta_k^e, z_k) \;, \end{aligned}$$

where e_{jk} comes from (2), and τ_{jk} is defined as in (7), and we consider e_{jk}, τ_{jk} as functions of (x_k, y_k, z_k). Notice that the inverse of relation (11) is

$$(12) \qquad \begin{aligned} \theta_k^f &= \tau_{kj}^{-1}(R\theta_j^{-1}, z_{2j}, z_{3j})\theta_j \;, & \theta_k^e &= \sigma_{kj}(R\theta_j^{-1}, z_{2j}, z_{3j}) \;, \\ r &= |e_{kj}(R\theta_j^{-1}, z_{2j}, z_{3j})|R \;, & s &= |f_{kj}(R\theta_j^{-1}, z_{2j}, z_{3j})| \;, \\ z_k &= h_{kj}(R\theta_j^{-1}, z_{2j}, z_{3j}) \;, \end{aligned}$$

where

$$(13) \qquad \begin{aligned} x_k &= e_{kj}(z_{1j}, z_{2j}, z_{3j})z_{1j} \;, \\ y_k &= f_{kj}(z_{1j}, z_{2j}, z_{3j}) \;, \\ z_k &= h_{kj}(z_{1j}, z_{2j}, z_{3j}) \;; \end{aligned}$$

$$e_{kj} = |e_{kj}|\tau_{kj} \;, \qquad f_{kj} = |f_{kj}|\sigma_{kj} \;.$$

Notice also that

$$(14) \qquad \tau_{kj}^e \cdot \sigma_{kj}^f = 1 \;.$$

Obviously (r, s, θ_k, z_k) uniquely determines $(R, \theta_j, z_{2j}, z_{3j})$ by (11). One should notice here that

$$\tau_{jk}^{-e}(r\theta_k^{-f}, s\theta_k^e, z_k)\theta_k^{ef} = 1 \;.$$

Conversely, by using a lemma similar to Lemma 2 in § 3 we see that $R(\theta_j, z_{2j}, z_{3j})$ uniquely determines (r, s, θ_k, z_k) by (12) and (14). Thus we get a space

$$\nu(\mathscr{N}^+) = \nu(\mathscr{U}^+) \cup \nu(\mathscr{W}^+) \cup \nu(\mathscr{V}^+) \;,$$

where $\mathscr{N} = \mathscr{U} \cup \mathscr{W} \cup \mathscr{V}$ and $\mathscr{N}^+ = \mathscr{M}^+ \cap \mathscr{N}$, and also have a map $\mu\colon \nu(\mathscr{N}^+) \to \mathscr{N}^+$ given for example by

$$\begin{aligned} \mu(R, \theta_j, z_{2j}, z_{3j}) &= (R\theta_j^{-1}, z_{2j}, z_{3j}) \;, \qquad && \text{on } \nu(\mathscr{U}_j^+) \;, \\ \mu(r, s, \theta_j, z_j) &= (r\theta_j^{-f}, s\theta_j^e, z_j) \;, && \text{on } \nu(\mathscr{W}_j^+) \;, \end{aligned}$$

where (x_j, y_j, z_j) are the coordinates on $\mathscr{W}_j^+$ defined before. Notice that $\nu(\mathscr{N}^+) - \mu^{-1}(M_0)$ is a differentiable manifold, which is in fact diffieomorphic to $\mathscr{N}^+$

$- M_0$ via the map μ which is a diffieomorphism on $\nu(\mathcal{N}^+) - \mu^{-1}(M_0) = \mu^{-1}(\mathcal{N}^+ - M_0)$. Thus we can define

$$\begin{aligned}\nu(\mathcal{M}^+) &= (\mathcal{M}^+ - M_0) \cup \nu(\mathcal{N}^+) \\ &= (\mathcal{M}^+ - M_0) \cup \mu^{-1}(M_0) , \qquad \text{(disjoint union)},\end{aligned}$$

where we use μ to identify $\nu(\mathcal{N}^+ - M_0)$ with $\mathcal{N}^+ - M_0 \subset \mathcal{M}^+ - M_0$. We can obviously extend μ and think of μ as a map $\mu\colon \nu(\mathcal{M}^+) \to \mathcal{M}^+$, where μ maps $\nu(\mathcal{M}^+ - M_0) = \nu(\mathcal{M}^+) - \mu^{-1}(M_0)$ diffeomorphically onto $\mathcal{M}^+ - M_0$. It is easy to see that $\nu(\mathcal{M}^+)$ is a topological manifold with boundary $\partial\nu(\mathcal{M}^+) = \mu^{-1}(M_0) = \nu(M_0)$.

We claim that

$$\nu(M_0) = \nu(M_0, \mathcal{H}_0) ,$$

where $\nu(M_0, \mathcal{H}_0)$ is the manifold constructed in § 3, the topological nonsingular model of the space $(M_0, \mathcal{H}_0)$. To prove this, set $U_j = M_0 \cap \mathcal{U}_j$, $V_j = M_0 \cap \mathcal{V}_j$, $W_j^1 = X \cap \mathcal{W}_j$ where $X \cap \mathcal{W}_j$ is given in terms of the coordinates (z_{1j}, z_{2j}, z_{3j}) for $\mathcal{W}_j$ by the equation $z_{1j} = 0$. If we compare equations (1), (2), and (3) of this section with equations (1), (2), and (3) of § 3 we see that (for example)

$$\begin{aligned}f_{jk}(z_{2k}, z_{3k}) &= f_{jk}(0, z_{2k}, z_{3k}) , &&\text{on } \mathcal{W}_j^1 \cap \mathcal{W}_k^1 , \\ e_{jk}(z_{2k}, z_{3k}) &= e_{jk}(0, z_{2k}, z_{3k}) , &&\text{on } \mathcal{W}_j^1 \cap \mathcal{W}_k^1 .\end{aligned} \tag{15}$$

For this choice of $f_{jk}(z_{2k}, z_{3k})$ and $e_{jk}(z_{2k}, z_{3k})$ we can choose the functions u_j of equation (4) in § 3 to be $u_j = 1$. Thus we can choose the v_j and β_1 to be $v_j = 1$, $\beta_j = 1$. We claim that there are maps

$$\begin{aligned}G&\colon \nu(M_0) - \mu^{-1}(Y) = \mu^{-1}(M_0 - Y) \to V_1 - R_1 , \\ H&\colon \nu(M_0) - \mu^{-1}(X) = \mu^{-1}(M_0 - X) \to V_2 - R_2 ,\end{aligned}$$

which are in fact homeomorphisms. For example

$$\mu^{-1}(M_0 - Y) \cap \nu(\mathcal{W}_j^+) = \{(0, s, \theta_j, z_j)\colon 0 < s < \varepsilon, \theta_j \in S^1, |z_j| < 1\} ,$$

and on this set

$$G(0, s, \theta_j, z_j) = ((s/b_j)^{f/e}\theta_j^f, (s/b_j)\theta_j^e, z_j) \in V_1 , \tag{16}$$

where $b_j = b_j(0, s\theta_j^e, z_j)$ and is considered as a function of (x_j, y_j, z_k). On $\mu^{-1}(M_0 - Y) \cap \nu(\mathcal{U}_j^+)$ we have

$$G(0, \theta_j, z_{2j}, z_{3j}) = (\theta_j, z_{2j}, z_{3j}) . \tag{17}$$

We can easily check that this map is a well-defined homeomorphism and in fact extends to a homeomorphism

$$G\colon \mu^{-1}(X) = \nu(X) \to J_1 ,$$

where $G\colon \nu(\bar{R}) \to \partial J_1$ is given by

$$G(0,0,\theta_j,z_j) = (0,\theta_j,z_j) \in \partial\hat{W}_j^1 \subset \partial J_1 , \tag{18}$$

and $(0,0,\theta_j,z_j) \in \nu(\bar{R} \cap \mathscr{W}_j^+)$. The analogous equations for H are

$$H(r,0,\theta_j,z_j) = ((r/a_j)^{e/f}\theta_j^{-e}, (r/a_j)\theta_j^{-f}, z_j) \in V_2 ,$$

$$H(0,0,\theta_j,z_j) = (0,\theta_j^{-1},z_j) \in \partial\hat{W}_j^2 \subset \partial J_2 . \tag{20}$$

Now consider equations (14) in § 3, which become $\varphi_j = \theta_j^{-1}$ since $\alpha_j = \beta_j = 1$. Let $\kappa\colon \partial J_1 \to \partial J_2$ denote this identification. Then $H = \kappa \circ G$ on $\nu(\bar{R})$. Since $\nu(M_0, \mathscr{H}_0) = J_1 \cup J_2$ where ∂J_1 is identified with ∂J_2 by κ, we see that H and G give rise to a homeomorphism between $\nu(M_0)$ and $\nu(M_0, \mathscr{H}_0)$.

The map $\bar{\omega}\mu\colon \nu(\mathscr{M}^+) \to \Delta^+$ satisfies

$$(\bar{\omega}\mu)^{-1}(t) = M_t \quad \text{for } t > 0 , \qquad (\bar{\omega}\mu)^{-1}(0) = \nu(M_0) .$$

Moreover $\nu(\mathscr{M}^+) - \nu(M_0)$ is a differentiable manifold. We will put a differentiable structure on $\nu(\mathscr{M}^+)$ which extends the differentiable structure on $\nu(\mathscr{M}^+) - \nu(M_0)$. We will have in this new structure a differentiable function with no critical points, which has as level sets the surfaces $\{(\bar{\omega}\mu)^{-1}(t)\colon t \geq 0\}$.

We define a C^∞ monotone increasing function as follows:

$$e(q) = \begin{cases} \exp(-1/q) & \text{for } q > 1 , \\ 0 & \text{for } q = 0 , \\ -\exp(1/q) & \text{for } q < 0 . \end{cases}$$

For $r, s \in \boldsymbol{R}^+ = \{t\colon t \geq 0\}$, the equations

$$e(\tau) = r^{1/f} \cdot s^{1/e} , \qquad 2e(q) = r^{1/f} - s^{1/e} \tag{21}$$

define a topological map $(r,s) \to (\tau, q)$ of a neighborhood of $(0,0)$ in $\boldsymbol{R}^+ \times \boldsymbol{R}^+$ to a neighborhood of $(0,0)$ in $\boldsymbol{R}^+ \times \boldsymbol{R}$. We easily check that the functions

$$\begin{aligned} r(\tau,q) &= (e(q) + \sqrt{e^2(q) + e(\tau)})^f , \\ s(\tau,q) &= (-e(q) + \sqrt{e^2(q) + e(\tau)})^e \end{aligned} \tag{22}$$

are C^∞ functions of q and τ in a neighborhood of $(0,0)$, and invert equations (21). Hence we may use (τ, q, θ_j, z_j) as new coordinates on $\nu(\mathscr{W}_j^+)$, so that (8) becomes

$$
(23) \qquad \begin{aligned}
\theta_j^f &= \tau_{jk}^{-1}(r(\tau, q)\theta_k^{-f}, s(\tau, q)\theta_k^e, z_k)\theta_k^f \ , \\
\theta_j^e &= \tau_{jk}(r(\tau, q)\theta_k^{-f}, s(\tau, q)\theta_k^e, z_k)\theta_k^e \ , \\
z_j &= h_{jk}(r(\tau, q)\theta_k^{-f}, s(\tau, q)\theta_k^e, z_k) \ .
\end{aligned}
$$

On $\nu(\mathscr{U}_j^+)$ we use the equation $e(\tau)^f = R$ to introduce $(\tau, \theta_j, z_{2j}, z_{3j})$ as new coordinates. As in (23) we easily check that (10), (11), and (12), due to (21) and (22), are turned into a differentiable change of coordinates between $(\tau, \theta_j, z_{2j}, z_{3j})$ and (τ, q, θ_k, z_k). In order to check this we need to assume that $\mathscr{U}_j$ is bounded away from $\bar{R} = X \cap Y$ so that we can verify that the Jacobian

$$
\det [\partial(\tau, z_{2j}, z_{3j}, \bar{z}_{2j}, \bar{z}_{3j}) / \partial(\tau, q, \theta_j, z_j, \bar{z}_j)]
$$

is bounded away from zero. We operate similarly on $\nu(\mathscr{V}_j^+)$. Thus we get a differentiable structure on $\nu(\mathscr{U}^+) \cup \nu(\mathscr{W}^+) \cup \nu(\mathscr{V}^+)$, which extends the given structure on $\nu(\mathscr{U}^+) \cup \nu(\mathscr{W}^+) \cup \nu(\mathscr{V}^+) - \nu(M_0)$. The function

$$
\tau = -ef/\log(\bar{\omega}\mu)
$$

is a differentiable function with no critial point on $\nu(\mathscr{M}^+)$, and

$$
\tau^{-1}(t) = \begin{cases} M_t & \text{if } t > 0 \ , \\ \nu(M_0) & \text{if } t = 0 \ . \end{cases}
$$

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